JOURNAL OF APPROXIMATION THEORY 23, 267–273 (1978)

On the Approximation of Functions and their Derivatives by Hermite Interpolation

PETER POTTINGER

Institut für Mathematik der Gesamthochschule Duisburg, Duisburg, West Germany

Communicated by Oved Shisha

Received December 16, 1976

1. INTRODUCTION

Given the interval $I = [-1, 1]$ and the space $C^k(I)$ consisting of the k-times continuously differentiable real-valued functions. Further, we provide $C^k(I)$ with the norm $\|\cdot\|_k$, which for a given $f \in C^k(I)$ is defined by

$$\|f\|_k := \max_{0 \leqslant \kappa \leqslant k} (\sup_{x \in I} |f^{(\kappa)}(x)|),$$

where $f^{(\kappa)}$ is the κth derivative of f.

For an arbitrary nodal matrix $M = \{x_0^n, \ldots, x_n^n\}_{n \in \mathbb{N}}$ we consider the Hermite interpolation operators (e.g., Natanson [2])

$$H_{2n+1} : C^1(I) \to C^1(I).$$

It is known that the convergence

$$\lim_{n \to \infty} \|f - H_{2n+1} f\|_1 = 0 \tag{1.1}$$

does not hold for each $f \in C^1(I)$ (cf. Esser and Scherer [1], Pottinger [4]). This raises the question for which classes of functions one can prove the convergence formula (1.1). We investigate this problem for the special case that the nodal matrix M consists of the Tchebycheff nodes. It turns out that the convergence property depends on the norms of the Hermite interpolation operators H_{2n+1}. Theorem 1 states that the growth of the operator norms is of order n. This estimation can not be improved (cf. [3]). With the aid of Theorem 1 we establish a convergence property in Theorem 2.

Some parts of the theory given in this paper have been established in [5].

0021-9045/78/0233–0267$02.00/0

Copyright © 1978 by Academic Press, Inc.
All rights of reproduction in any form reserved.

2. Some Estimations and Convergence Properties

In the following we take as interpolation nodes the roots $x_\mu = \cos \Theta_\mu$[1] with $\Theta_\mu = ((2\mu + 1)/ 2 \cdot (n + 1)) \cdot \pi\ (0 \leqslant \mu \leqslant n)$ of the Tchebycheff polynomials. Then the Hermite interpolation operators $H_{2n+1} : C^1(I) \to C^1(I)$ are defined by (e.g., Natanson [2])

$$H_{2n+1}f(x) := \sum_{\mu=0}^{n} v_\mu(x) \cdot l_\mu^2(x) \cdot f(x_\mu)$$

$$+ \sum_{\mu=0}^{n} (x - x_\mu) \cdot l_\mu^2(x) \cdot f'(x_\mu) \qquad (f \in C^1(I))$$

with

$$v_\mu(x) := 1 - \frac{\cos \Theta_\mu}{\sin^2 \Theta_\mu} \cdot (x - x_\mu)$$

and

$$l_\mu(x) := \frac{(-1)^\mu}{n+1} \cdot \frac{\cos(n+1)\,\Theta \cdot \sin \Theta_\mu}{\cos \Theta - \cos \Theta_\mu} \qquad (x = \cos \Theta, 0 \leqslant \Theta \leqslant \pi).$$

We first have to prove some estimations. To this end, we define the continuous functions A_n, B_n, and C_n for $x \in I$ by

$$A_n(x) := \sum_{\mu=0}^{n} \frac{|\sin(n+1)\,\Theta|}{\sin \Theta} \cdot \sin \Theta_\mu \cdot |l_\mu(x)|,$$

$$B_n(x) := \sum_{\mu=0}^{n} \frac{|\cos \Theta_\mu|}{\sin^2 \Theta_\mu} \cdot |x - x_\mu| \cdot l_\mu^2(x) \qquad (x = \cos \Theta, 0 \leqslant \Theta \leqslant \pi),$$

$$C_n(x) := \sum_{\mu=0}^{n} \frac{|\sin(n+1)\,\Theta|}{\sin \Theta} \cdot \frac{|\cos \Theta_\mu|}{\sin \Theta_\mu} \cdot |x - x_\mu| \cdot |l_\mu(x)|.$$

Lemma 1. *The following estimations hold true, when n runs to infinity*:

(a) $\| A_n \|_0 = O(n)$,

(b) $\| B_n \|_0 = O(\log n)$,

(c) $\| C_n \|_0 = O(n)$.

Proof. Using the formula for l_μ and the estimation $\| \sum_{\mu=0}^{n} | l_\mu | \|_0 = O(\log n)$ (e.g., Natanson [2]) one can easily prove the parts (b) and (c). To

[1] We will omit the upper index "*n*."

derive the estimation for A_n we first consider the case that we have sin $\Theta \geqslant n^{-1/2}$ ($x = \cos\Theta$, $0 \leqslant \Theta \leqslant \pi$). Then we obtain

$$A_n(x) \leqslant n^{1/2} \sum_{\mu=0}^{n} |l_\mu(x)| \leqslant c \cdot n^{1/2} \cdot \log(n+1) \qquad \text{(e.g. Natanson [2])},$$

where the constant c does not depend on x. For $\sin\Theta < n^{-1/2}$ we get

$$A_n(x_\mu) = 1$$

and

$$A_n(x) \leqslant \sum_{\mu=0}^{n} (n+1) \cdot \sin\Theta_\mu \cdot |l_\mu(x)| \qquad (\Theta \neq \Theta_\mu)$$

$$= \sum_{\mu=0}^{n} \frac{\sin^2\Theta_\mu \cdot |\cos(n+1)\Theta|}{|\cos\Theta - \cos\Theta_\mu|}$$

$$\leqslant \sum_{\mu=0}^{n} \frac{\sin\Theta_\mu \cdot |\sin\Theta - \sin\Theta_\mu| \cdot |\cos(n+1)\Theta|}{|\cos\Theta - \cos\Theta_\mu|}$$

$$+ \sum_{\mu=0}^{n} \frac{\sin\Theta_\mu \cdot \sin\Theta \cdot |\cos(n+1)\Theta|}{|\cos\Theta - \cos\Theta_\mu|}$$

$$=: \bar{A}_n^1(x) + \bar{A}_n^2(x).$$

This yields

$$\bar{A}_n^2(x) = \sum_{\mu=0}^{n} \frac{\sin\Theta_\mu \cdot \sin\Theta \cdot |\cos(n+1)\Theta|}{|\cos\Theta - \cos\Theta_\mu|}$$

$$\leqslant \frac{n+1}{n^{1/2}} \cdot \sum_{\mu=0}^{n} |l_\mu(x)| \leqslant d \cdot n^{1/2} \cdot \log(n+1)$$

with a constant d, which is independent of x.

For $\bar{A}_n^1$ we get

$$\bar{A}_n^1(x) \leqslant \sum_{\mu=0}^{n} \frac{(\sin\Theta_\mu + \sin\Theta) \cdot |\sin\Theta_\mu - \sin\Theta|}{|\cos\Theta - \cos\Theta_\mu|}$$

$$= 2 \cdot \sum_{\mu=0}^{n} \left| \frac{\sin\frac{\Theta+\Theta_\mu}{2} \cdot \sin\frac{\Theta-\Theta_\mu}{2} \cdot \cos\frac{\Theta+\Theta_\mu}{2} \cdot \cos\frac{\Theta-\Theta_\mu}{2}}{\sin\frac{\Theta+\Theta_\mu}{2} \cdot \sin\frac{\Theta-\Theta_\mu}{2}} \right|$$

$$\leqslant 2\, n + 1),$$

what concludes our proof. □

By $\| H_{2n+1} \|$ we denote the operator norm of H_{2n+1}, which belongs to the given $\| \cdot \|_1$ on $C^1(I)$. In [3] it was proved that $\| H_{2n+1} \| \geqslant 2n - 4$. Now we derive an upper bound for $\| H_{2n+1} \|$:

THEOREM 1. *The estimation*

$$\| H_{2n+1} \| = O(n) \qquad (n \to \infty)$$

holds true.

Proof. For $f \in C^1(I)$ with $\|f\|_1 = 1$ one easily veryfies

$$\| H_{2n+1} f \|_0 \leqslant 5.$$

Because of

$$\sum_{\mu=0}^{n} v_\mu(x) \cdot l_\mu^2(x) = 1, \qquad \text{for each } x \in I,$$

we get for $f \in C^1(I)$

$$\begin{aligned} H_{2n+1} f(x) - f(x) &= \sum_{\mu=0}^{n} v_\mu(x) \cdot l_\mu^2(x) \cdot (f(x_\mu) - f(x)) \\ &\quad + \sum_{\mu=0}^{n} (x - x_\mu) \cdot l_\mu^2(x) \cdot f'(x_\mu) \\ &= \sum_{\mu=0}^{n} v_\mu(x) \cdot l_\mu^2(x) \cdot \left(\int_x^{x_\mu} f'(t)\, dt \right) \\ &\quad + \sum_{\mu=0}^{n} (x - x_\mu) \cdot l_\mu^2(x) \cdot f'(x_\mu). \end{aligned}$$

By differentiation we obtain for $f \in C^1(I)$ with $\|f\|_1 = 1$

$$\begin{aligned} (H_{2n+1} f)'(x) &= \sum_{\mu=0}^{n} v_\mu'(x) \cdot l_\mu^2(x) \cdot \left(\int_x^{x_\mu} f'(t)\, dt \right) \\ &\quad + 2 \cdot \sum_{\mu=0}^{n} v_\mu(x) \cdot l_\mu(x) \cdot l_\mu'(x) \cdot \left(\int_x^{x_\mu} f'(t)\, dt \right) \\ &\quad + 2 \cdot \sum_{\mu=0}^{n} (x - x_\mu) \cdot l_\mu(x) \cdot l_\mu'(x) \cdot f'(x_\mu) \\ &\quad + \sum_{\mu=0}^{n} l_\mu^2(x) \cdot f'(x_\mu) \end{aligned}$$

and

$$|(H_{2n+1}f)'(x)| \leqslant \sum_{\mu=0}^{n} | v_\mu'(x) \cdot l_\mu^2(x) \cdot (x - x_\mu)|$$

$$+ 2 \cdot \sum_{\mu=0}^{n} | v_\mu(x) \cdot l_\mu(x) \cdot l_\mu'(x) \cdot (x - x_\mu)|$$

$$+ 2 \cdot \sum_{\mu=0}^{n} |(x - x_\mu) \cdot l_\mu(x) \cdot l_\mu'(x)| + \sum_{\mu=0}^{n} l_\mu^2(x).$$

Further, we have

$$\sum_{\mu=0}^{n} | v_\mu'(x) \cdot l_\mu^2(x) \cdot (x - x_\mu)| = B_n(x) \qquad \text{(cf. Lemma 1).}$$

Because of

$$l_\mu'(x) = \frac{1}{\cos \Theta - \cos \Theta_\mu} \cdot \left((-1)^\mu \cdot \sin \Theta_\mu \cdot \frac{\sin(n+1)\,\Theta}{\sin \Theta} - l_\mu(x)\right),$$

$$(x = \cos \Theta,\ 0 \leqslant \Theta \leqslant \pi),$$

we obtain

$$\sum_{\mu=0}^{n} | v_\mu(x) \cdot l_\mu(x) \cdot l_\mu'(x) \cdot (x - x_\mu)|$$

$$\leqslant \sum_{\mu=0}^{n} |(x - x_\mu) \cdot l_\mu(x) \cdot l_\mu'(x)|$$

$$+ \sum_{\mu=0}^{n} \left| \frac{\cos \Theta_\mu}{\sin^2 \Theta_\mu} \cdot (x - x_\mu)^2 \cdot l_\mu(x) \cdot l_\mu'(x)\right|$$

$$\leqslant 2 \cdot \sum_{\mu=0}^{n} \frac{| \sin(n+1)\,\Theta |}{\sin \Theta} \cdot \sin \Theta_\mu \cdot | l_\mu(x)| + 2 \cdot \sum_{\mu=0}^{n} l_\mu^2(x)$$

$$+ \sum_{\mu=0}^{n} \frac{| \cos \Theta_\mu |}{\sin^2 \Theta_\mu} \cdot | x - x_\mu | \cdot l_\mu^2(x)$$

$$+ \sum_{\mu=0}^{n} \left| \frac{\sin(n+1)\,\Theta}{\sin \Theta} \cdot \frac{\cos \Theta_\mu}{\sin \Theta_\mu} \cdot (x - x_\mu) \cdot l_\mu(x)\right|$$

$$= 2 \cdot A_n(x) + B_n(x) + C_n(x) + 2 \cdot \sum_{\mu=0}^{n} l_\mu^2(x) \qquad \text{(cf. Lemma 1)}$$

and—as was proved above—

$$\sum_{\mu=0}^{n} |(x - x_\mu) \cdot l_\mu(x) \cdot l'_\mu(x)| \leqslant A_n(x) + \sum_{\mu=0}^{n} l_\mu^2(x).$$

Summarizing these estimations we get with the aid of Lemma 1

$$\| H_{2n+1} \|_1 = O(n) \qquad (n \to \infty),$$

since $\sum_{\mu=0}^{n} l_\mu^2(x) \leqslant 2$. □

For a given $f \in C^1(I)$ we define the approximation constants $E_n(f)$ and $E_n^1(f)$ by

$$E_n(f) := \inf_{\pi \in \Pi_n} \| f - \pi \|_0 , \qquad E_n^1(f) := \inf_{\pi \in \Pi_n} \| f - \pi \|_1 ,$$

where Π_n is the space polynomials of degree $\leqslant n$. Further, we get

$$E_n^1(f) = E_{n-1}(f').$$

Because of the estimation

$$\| f - H_{2n+1} f \|_1 \leqslant (\| H_{2n+1} \| + 1) \cdot E_{2n+1}^1(f)$$

we obtain the following convergence property:

THEOREM 2. (a) *For a given* $f \in C^2(I)$ *we have*

$$\lim_{n \to \infty} \| f - H_{2n+1} f \|_1 = 0.$$

(b) *If* $f \in C^k(I)$ $(k \geqslant 3)$, *we get*

$$\| f - H_{2n+1} f \|_1 = O\left(\frac{1}{n^{k-2}}\right) \qquad (n \to \infty).$$

(c) *For* $f \in C^k(I)$ $(k \geqslant 2)$ *with* $f^{(k)} \in \mathrm{Lip}\, \alpha (0 < \alpha \leqslant 1)$ *we obtain*

$$\| f - H_{2n+1} f \|_1 = O\left(\frac{1}{n^{k+\alpha-2}}\right) \qquad (n \to \infty).$$

ACKNOWLEDGMENT

I wish to express my gratitude to Professor W. Haußmann, Duisburg, for his constant interest in my work and for his helpful suggestions.

REFERENCES

1. H. ESSER AND K. SCHERER, Eine Bemerkung zur Konvergenz Hermitescher Interpolationsprozesse, *Numer. Math.* **21** (1973), 220–222.
2. I. P. NATANSON, "Constructive Function Theory," Vol. III, Ungar, New York, 1965.
3. P. POTTINGER, Zur Hermite-Interpolation, *Z. Angew. Math. Mech.* **56** (1976), T310–T311.
4. P. POTTINGER, Polynomoperatoren in $C^r[a, b]$, Computing **17** (1976), 163–167.
5. P. POTTINGER, "Zur linearen Approximation im Raum $C^k(I)$," Habilitationsschrift, Duisburg, 1976.